On perfect 2-colorings of the q-ary n-cube *

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Abstract

A coloring of the q-ary n-dimensional cube (hypercube) is called perfect if, for every n-tuple x, the collection of the colors of the neighbors of x depends only on the color of x. A Boolean-valued function is called correlation-immune of degree $n-m$ if it takes the value 1 the same number of times for each m-dimensional face of the hypercube. Let $f=\chi^S$ be a characteristic function of some subset S of hypercube. In the present paper it is proven the inequality $\rho(S)q(\mathrm{cor}(f)+1)\leq\alpha(S)$, where $\mathrm{cor}(f)$ is the maximum degree of the correlation immunity of f, $\alpha(S)$ is the average number of neighbors in the set S for n-tuples in the complement of a set S, and $\rho(S)=|S|/q^n$ is the density of the set S. Moreover, the function f is a perfect coloring if and only if we obtain an equality in the above formula. Also we find new lower bound for the cardinality of components of perfect coloring and 1-perfect code in the case $q>2$.

Keywords: hypercube, perfect coloring, perfect code, MDS code, bitrade, equitable partition, orthogonal array.

1. Introduction

Let Z_q be the set of entries $\{0,\dots,q-1\}$. The set Z_q^n of n-tuples of entries is called q-ary n-dimensional cube (hypercube). The *Hamming distance* $d(x,y)$ between two n-tuples $x,y\in Z_q^n$ is the number of positions at which they differ. Define the number $\alpha(S)$ to be the average number of neighbors in the set $S\subseteq Z_q^n$ for n-tuples in the complement of a set S, i. e., $\alpha(S)=\frac{1}{q^n-|S|}\sum\limits_{x\notin S}|\{y\in S\mid d(x,y)=1\}|$.

A mapping $Col: Z_q^n\to\{0,\dots,k\}$ is called a *perfect coloring* with matrix of parameters $A=\{a_{ij}\}$ if, for all i, j, for every n-tuple of color i, the number of its neighbors of color j is equal to a_{ij}. Other terms used for this notion in the literature are "equitable partition", "partition design" and "distributive coloring". In what follows we will only consider colorings in two colors (2-coloring). Moreover, for convenience we will assume that the set of colors is $\{0,1\}$. In this case the Boolean-valued function Col is a characteristic function of the set of 1-colored n-tuples.

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A *1-perfect code* (one-error-correcting) $C \subset Z_q^n$ can be defined as the set of units of a perfect coloring with matrix of parameters $A = \begin{pmatrix} n(q-1)-1 & 1 \\ n(q-1) & 0 \end{pmatrix}$. If q is the power of a prime number then the coloring with such parameters exists only if $n = \frac{q^m-1}{q-1}$ (m is integer). For $q=2$ a list of accessible parameters and corresponding constructions of perfect 2-colorings can be found in [1] and [2].

In [3] it is established that for each unbalanced Boolean function $f = \chi^S$ ($S \subset Z_2^n$) the inequality $\mathrm{cor}(f) \le \frac{2n}{3} - 1$ holds. Moreover, in the case of the equality $\mathrm{cor}(f) = \frac{2n}{3} - 1$, the function f is a perfect 2-coloring. Similarly, if for any set $S \subset Z_2^n$ the Friedman (see [4]) inequality $\rho(S) \ge 1 - \frac{n}{2(\mathrm{cor}(f)+1)}$ becomes an equality then the function χ^S is a perfect 2-coloring (see [6]). Consequently, in the extremal cases, the regular distribution on balls follows from the uniform distribution on faces. The main result of present paper is following theorem:

Theorem 1.
(a) *For each Boolean-valued function $f = \chi^S$, where $S \subset Z_q^n$, the inequality $\rho(S)q(\mathrm{cor}(f)+1) \le \alpha(S)$ holds.*
(b) *A Boolean-valued function $f = \chi^S$ is a perfect 2-coloring if and only if $\rho(S)q(\mathrm{cor}(f)+1) = \alpha(S)$.*

2. Criterion for perfect 2-coloring

In the proof of the theorem we employ the idea from the papers [5].

Now we consider Z_q as the cyclic group on the set of entries $\{0,\dots,q-1\}$. We may impose the structure of the group $Z_q \times \dots \times Z_q$ on the hypercube. Consider the vector space $\mathbb{V}$ of complex-valued function on Z_q^n with scalar product $(f,g) = \frac{1}{q^n} \sum_{x \in Z_q^n} f(x)\overline{g(x)}$. For every $z \in Z_q^n$ define a *character* $\phi_z(x) = \xi^{\langle x,z\rangle}$, where $\xi = e^{2\pi i/q}$ is a primitive complex qth root of unity and $\langle x,z\rangle = x_1z_1 + \dots + x_nz_n$. Here all arithmetic operations are performed on complex numbers. As is generally known the characters of the group $Z_q \times \dots \times Z_q$ form an orthonormal basis of $\mathbb{V}$. It is sufficient to verify that $\xi^k\overline{\xi^k} = 1$ and $\sum_{j=0}^{q-1} \xi^{kj} = 0$ as $k \ne 0 \mod q$.

Let M be the adjacency matrix by the hypercube Z_q^n. This means that $Mf(x) = \sum_{y, d(x,y)=1} f(y)$. It is well known that the characters are eigenvectors of M. Indeed we have

$$M\phi_z(x) = \sum_{y,d(x,y)=1} \xi^{\langle y-x,z\rangle + \langle x,z\rangle} = \xi^{\langle x,z\rangle} \sum_{j=1}^{n} \sum_{k\ne 0} \xi^{kz_j} = ((n - wt(z))(q-1) - wt(z))\phi_z(x),$$

where $wt(z)$ is the number of nonzero coordinates of z.

Consider a perfect coloring $f \in \mathbb{V}$, $f(Z_q^n) = \{0,1\}$ with matrix of parameters

$$A = \begin{pmatrix} n(q-1)-b & b \\ c & n(q-1)-c \end{pmatrix}. \tag{1}$$

The vector $(-b, c)$ is an eigenvector of A with the eigenvalue $n(q-1)-c-b$. The definition of a perfect 2-coloring implies that the function $(b+c)f-b$ is the eigenvector of matrix M. Moreover the converse is true: every two-valued eigenvector of matrix M generates a perfect coloring.

Proposition 1. *(see [1])*

(a) *Let f be a perfect 2-coloring with matrix of parameters A (1). Then $s = \frac{c+b}{q}$ is integer and $(f, \phi_z) = 0$ for every n-tuples $z \in Z_q^n$ such that $wt(z) \neq 0, s$.*

(b) *Let $f : Z_q^n \to \{0,1\}$ be a Boolean-valued function. If $(f, \phi_z) = 0$ for every n-tuples $z \in \{0, \dots, q-1\}^n$ such that $wt(z) \neq 0, s$ then f is a perfect 2-coloring.*

Refer as a *correlation-immune* function of order $n-m$ to a function $f \in \mathbb{V}$ that every value is uniformly distributed on all m-dimensional faces. For any function $f \in \mathbb{V}$ we denote the maximum of order of its correlation-immunity by $\mathrm{cor}(f)$. Consider a nonempty set of n-tuples $O(a) = f^{-1}(a) \subset Z_q^n$ where $a \in \mathbb{C}$. An array consisted of n-tuples $x \in O(a)$ is called *orthogonal array* with parameters $OA_\lambda(\mathrm{cor}(f), n, q)$, where $\lambda = |O(a)|/q^{n-\mathrm{cor}(f)}$.

Proposition 2. *(see [5])*

(a) *If $f \in \mathbb{V}$ is a correlation-immune function of order m then $(f, \phi_z) = 0$ for every n-tuples $z \in Z_q^n$ such that $0 < wt(z) \leq m$.*

(b) *A Boolean-valued function $f \in \mathbb{V}$ is correlation-immune of order m if $(f, \phi_z) = 0$ for every n-tuples $z \in Z_q^n$ such that $0 < wt(z) \leq m$.*

Corollary 1. *Let f be a perfect 2-coloring with matrix of parameters (1). Then $\mathrm{cor}(f) = \frac{c+b}{q} - 1$.*

For 1-perfect codes last statement was proven otherwise in [7].

Proof of the theorem. We have the following equalities by the definitions and general properties of orthonormal basis.

$$\sum_z |(f, \phi_z)|^2 = \frac{1}{q^n} \sum_{x \in Z_q^n} |f(x)|^2 = \rho(S). \tag{2}$$

$$(f, \phi_{\overline{0}}) = \frac{1}{q^n} \sum_{x \in Z_q^n} f(x) = \rho(S). \tag{3}$$

$$(Mf, f) = \frac{1}{q^n} \sum_{x \in Z_q^n} \sum_{y, d(x,y)=1} f(x)\overline{f(y)} = \mathrm{nei}(S)\rho(S), \tag{4}$$

where $\mathrm{nei}(S) = \frac{1}{|S|} \sum\limits_{x \in S} |\{y \in S \mid d(x,y) = 1\}|$.

$$(Mf, f) = \sum_{z \in Z_q^n} (n(q-1) - wt(z)q)|(f, \phi_z)|^2. \tag{5}$$

From (2–5) and Proposition 2 we obtain the equality

$$\mathrm{nei}(S)\rho(S) = \rho(S)^2 n(q-1) + \sum_{z, wt(z) \geq \mathrm{cor}(f)+1} (n(q-1) - wt(z)q)|(f, \phi_z)|^2.$$

Since $\sum\limits_{z, wt(z) \geq \mathrm{cor}(f)+1} |(f, \phi_z)|^2 = \rho(S) - \rho(S)^2$, we have

$$\mathrm{nei}(S)\rho(S) \leq \rho(S)^2 n(q-1) + (n(q-1) - (\mathrm{cor}(f)+1)q)(\rho(S) - \rho(S)^2) \text{ and}$$

$$(\mathrm{cor}(f)+1)q(1-\rho(S)) \leq n(q-1) - \mathrm{nei}(S). \tag{6}$$

Substitute the set $Z_q^n \setminus S$ instead of the set S into the inequality (6). Since $\mathrm{cor}(\chi^S) = \mathrm{cor}(\chi^{Z_q^n \setminus S})$, $1 - \rho(Z_q^n \setminus S) = \rho(S)$ and $n(q-1) - \mathrm{nei}(Z_q^n \setminus S) = \alpha(S)$ we obtain the item (a) of the Theorem.

Moreover, the equality

$$(\mathrm{cor}(f)+1)q(1-\rho(S)) = n(q-1) - \mathrm{nei}(S) \tag{7}$$

holds if and only if $(f, \phi_z) = 0$ for every n-tuple z such that $wt(z) \geq \mathrm{cor}(f) + 2$. Then from Proposition 1 (b) we conclude that f is a perfect 2-coloring.

Each perfect 2-coloring satisfies (7), which is a consequence of Proposition 1 (a) and Corollary 1. As mentioned above the equality (7) is equivalent to the equality in the item (b) of the Theorem. $\square$

Since $\mathrm{nei}(S) \neq 0$, the inequality (6) implies the Bierbrauer – Friedman inequality (see [4], [5])

$$\rho(S) \geq 1 - \frac{n(q-1)}{q(\mathrm{cor}(f)+1)}.$$

For 1-perfect binary codes, a similar theorem was previously proven in [10]. Namely, if $\mathrm{cor}(S) = \mathrm{cor}(H)$ and $\rho(S) = \rho(H)$, where $S, H \subset Z_2^n$ and H is a 1-perfect code, then S is also a 1-perfect code.

3. Components of perfect 2-coloring

Refer as a *bitrade* of order $n - m$ to a subset $B \subseteq Z_q^n$ that the cardinality of intersections S and each m-dimensional face are even.

Proposition 3. *Let $S \subseteq Z_2^n$ be a nonempty bitrade of order m. Then $|S| \geq 2^{m+1}$.*

Proposition 3 formulated in other term was proven in [11].

Proposition 4. *Let $S \subseteq Z_q^n$ $(q > 2)$ be a nonempty bitrade of order m. Then $|S| \geq 2^{m+1}$.*

Proof. Suppose that this statement is true for $n = k$. We will prove it for $n = k+1$. Let there exist three parallel k-dimensional faces F_1, F_2, F_3 such that intersections $F_i \cap S$ are nonempty. By induction hypothesis $|F_i \cap S| \geq 2^{m-1}$ for $i = 1, 2, 3$; consequently, $|S| \geq 3 \cdot 2^{m-1}$. In the other case $|S| \geq 2^m$ by Proposition 3. $\square$

Let characteristic functions $f = \chi^{S_1}$ and $g = \chi^{S_2}$ be perfect 2-colorings (correlation-immune) with an equal matrix of the parameters ($\mathrm{cor}(f) = \mathrm{cor}(g)$). A set $S_1 \triangle S_2$ is called *mobile* and sets $S_1 \setminus S_2$ and $S_2 \setminus S_1$ are called *components* of a perfect 2-colorings (correlation-immune functions) χ^{S_1} and χ^{S_2} respectively. It is clear, that a mobile set of correlation-immune function of order m is a bitrade of order m.

Corollary 2.

(a) Let f be a perfect 2-coloring with matrix of parameters (1). If $S \subset Z_q^n$ is a component of f then $|S| \geq 2^{\frac{c+b}{q}-1}$.

b) Let $C \subset Z_p^n$ be a 1-perfect code. If $S \subset Z_q^n$ is a component of f then $|S| \geq 2^{\frac{n(q-1)+1}{q}-1}$.

If $q = 2$ then the lower bound $|S| \geq 2^{\frac{n+1}{2}-1}$ for the cardinality of components of 1-perfect codes is achievable (see, for example, [6]). In the case $q > 2$ an upper bound for the cardinality of components of 1-perfect codes is obtained constructively (see [8], [9]). If $q = p^r$ and p is a prime number then $|S| \geq p^{\frac{q^{m-1}-1}{q-1}(r(q-2)+1)}$ where $n = \frac{q^m-1}{q-1}$.

A set $S \subset Z_p^n$ is called MDS code with distance 2 if intersection S and each 1-dimensional face contains precisely one n-tuple. Obviously a characteristic function of MDS code is a perfect 2-coloring with matrix of parameters $\begin{pmatrix} n(q-2) & n \\ n(q-1) & 0 \end{pmatrix}$. If $q \geq 4$ then the lower bound $|S| \geq 2^{n-1}$ for the cardinality of the components of MDS codes is achievable (see [12]).

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